BRACE OPERATIONS AND DELIGNE'S CONJECTURE FOR MODULE-ALGEBRAS

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A . It is observed that Kaygun's Hopf-Hochschild cochain complex for a module-algebra is a brace algebra with multiplication. As a result, (i) an analogue of Deligne's Conjecture holds for module-algebras, and (ii) the Hopf-Hochschild cohomology of a module-algebra has a Gerstenhaber algebra structure.

1. I

Let H be a bialgebra and let A be an associative algebra. The algebra A is said to be an H-module-algebra if there is an H-module structure on A such that the multiplication on A becomes an H-module morphism. For example, if S denotes the Landweber-Novikov algebra [15, 21], then the complex cobordism $MU^*(X)$ of a topological space X is an S-module-algebra. Likewise, the singular mod p cohomology $H^*(X; \mathbf{F}_p)$ of a topological space X is an $\mathcal{A}_p$-module-algebra, where $\mathcal{A}_p$ denotes the Steenrod algebra associated to the prime p [7, 19]. Other similar examples from algebraic topology can be found in [4]. Important examples of module-algebras from Lie and Hopf algebras theory can be found in, e.g., [12, V.6].

In [14], Kaygun defined a Hochschild-like cochain complex $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ associated to an H-module-algebra A, called the *Hopf-Hochschild cochain complex*, that takes into account the H-linearity. In particular, if H is the ground field, then Kaygun's Hopf-Hochschild cochain complex reduces to the usual Hochschild cochain complex $C^*(A, A)$ of A [11]. Kaygun [14] showed that the Hopf-Hochschild cohomology of A shares many properties with the usual Hochschild cohomology. For example, it can be described in terms of derived functors, and it satisfies Morita invariance.

The usual Hochschild cochain complex $C^*(A, A)$ has a very rich structure. Namely, it is a brace algebra with multiplication [10]. Combined with a result of McClure and Smith [18] concerning the singular chain operad associated to the little squares operad $\mathcal{C}_2$, the brace algebra with multiplication structure on $C^*(A, A)$ leads to a positive solution of Deligne's Conjecture [6]. Also, passing to cohomology, the brace algebra with multiplication structure implies that the Hochschild cohomology modules $HH^*(A, A)$ form a

Gerstenhaber algebra, which is a graded version of a Poisson algebra. This fact was first observed by Gerstenhaber [8].

The purpose of this note is to observe that Kaygun's Hopf-Hochschild cochain complex $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ of a module-algebra A also admits the structure of a brace algebra with multiplication. As in the classical case, this leads to a version of Deligne's Conjecture for module-algebras. Also, the Hopf-Hochschild cohomology modules $HH^*_{\mathrm{Hopf}}(A, A)$ form a Gerstenhaber algebra. When the bialgebra H is the ground field, these structures reduce to the ones in Hochschild cohomology.

A couple of remarks are in order. First, there is another cochain complex $\mathcal{F}^*(A)$ that can be associated to an H-module-algebra A [23]. The cochain complex $\mathcal{F}^*(A)$ is a differential graded algebra. Moreover, it controls the deformations of A, in the sense of Gerstenhaber [9], with respect to the H-module structure, leaving the algebra structure on A fixed. It is not yet known whether $\mathcal{F}^*(A)$ is a brace algebra with multiplication and whether the cohomology modules of $\mathcal{F}^*(A)$ form a Gerstenhaber algebra.

Second, the results and arguments here can be adapted to module-coalgebras, comodule-algebras, and comodule-coalgebras. To do that, one replaces the crossed product algebra X (§2.3) associated to an H-module-algebra A by a suitable crossed product (co)algebra [1, 2, 3] and replaces Kaygun's Hopf-Hochschild cochain complex by a suitable variant.

1.1. **Organization.** The rest of this paper is organized as follows.

In the following section, we recall the construction of the Hopf-Hochschild cochain complex $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ from Kaygun [14]. In Section 3, it is observed that $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ has the structure of an operad with multiplication (Theorem 3.4). This leads in Section 4 to the desired brace algebra with multiplication structure on $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ (Corollary 4.4). Explicit formulas for the brace operations are given.

In Section 5, it is observed that the brace algebra with multiplication structure on $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ leads to a homotopy G-algebra structure (Corollary 5.3). The differential from this homotopy G-algebra and the Hopf-Hochschild differential are then identified, up to a sign (Theorem 5.5). Passing to cohomology, this leads in Section 6 to a Gerstenhaber algebra structure on the Hopf-Hochschild cohomology modules $HH^*_{\mathrm{Hopf}}(A, A)$ (Corollary 6.3). The graded associative product and the graded Lie bracket on $HH^*_{\mathrm{Hopf}}(A, A)$ are explicitly described.

In the final section, by combining our results with a result of McClure and Smith [18], a version of Deligne's Conjecture for module-algebras is obtained (Corollary 7.1). This section can be read immediately after Section 4 and is independent of Sections 5 and 6.

2. H -H

In this section, we fix some notations and recall from [14, Section 3] the Hopf-Hochschild cochain complex associated to a module-algebra.

2.1. **Notations.** Fix a ground field K once and for all. Tensor product and vector space are all meant over K.

Let $H = (H, \mu_H, \Delta_H)$ denote a K-bialgebra with associative multiplication μ_H and coassociative comultiplication Δ_H. It is assumed to be unital and counital, with its unit and counit denoted by 1_H and $\varepsilon\colon H \to K$, respectively.

Let $A = (A, \mu_A)$ denote an associative, unital K-algebra with unit 1_A (or simply 1).

In a coalgebra (C, Δ), we use Sweedler's notation [22] for comultiplication: $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$, $\Delta^2(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$, etc.

These notations will be used throughout the rest of this paper.

2.2. **Module-algebra.** Recall that the algebra A is said to be an *H-module-algebra* [5, 12, 20, 22] if and only if there exists an H-module structure on A such that μ_A is an H-module morphism, i.e.,

$$b(a_1a_2) \;=\; \sum (b_{(1)}a_1)(b_{(2)}a_2) \tag{2.2.1}$$

for $b \in H$ and $a_1, a_2 \in A$. It is also assumed that $b(1_A) = \varepsilon(b)1_A$ for $b \in H$.

We will assume that A is an H-module-algebra for the rest of this paper.

2.3. **Crossed product algebra.** Let X be the vector space $A \otimes A \otimes H$. Define a multiplication on X [14, Definition 3.1] by setting

$$(a_1 \otimes a_1' \otimes b^1)(a_2 \otimes a_2' \otimes b^2) \;\stackrel{\text{def}}{=}\; \sum a_1 \left(b^1_{(1)}a_2\right) \otimes \left(b^1_{(3)}a_2'\right) a_1' \otimes b^1_{(2)}b^2 \tag{2.3.1}$$

for $a_1 \otimes a_1' \otimes b^1$ and $a_2 \otimes a_2' \otimes b^2$ in X. It is shown in [14, Lemma 3.2] that X is an associative, unital K-algebra, called the *crossed product algebra*.

Note that if $H = K$ (= the trivial group bialgebra $K[\{e\}]$), then X is just the enveloping algebra $A \otimes A^{\mathrm{op}}$, where A^{op} is the opposite algebra of A.

The algebra A is an X-module via the action

$$(a \otimes a' \otimes b)a_0 \;=\; a(ba_0)a'$$

for $a \otimes a' \otimes b \in X$ and $a_0 \in A$. Likewise, the vector space $A^{\otimes(n+2)}$ is an X-module via the action

$$(a \otimes a' \otimes b)(a_0 \otimes \cdots \otimes a_{n+1}) \;=\; \sum ab_{(1)}a_0 \otimes b_{(2)}a_1 \otimes \cdots \otimes b_{(n+1)}a_n \otimes b_{(n+2)}a_{n+1}a'$$

for $a_0 \otimes \cdots \otimes a_{n+1} \in A^{\otimes(n+2)}$.

2.4. **Bar complex.** Consider the chain complex $\mathrm{CB}_*(A)$ of vector spaces with $\mathrm{CB}_n(A) = A^{\otimes(n+2)}$, whose differential $d_n^{\mathrm{CB}} \colon \mathrm{CB}_n(A) \to \mathrm{CB}_{n-1}(A)$ is defined as the alternating sum $d_n^{\mathrm{CB}} = \sum_{j=0}^n (-1)^j \partial_j$, where

$$\partial_j(a_0 \otimes \cdots \otimes a_{n+1}) \;=\; a_0 \otimes \cdots \otimes (a_j a_{j+1}) \otimes \cdots \otimes a_{n+1}.$$

It is mentioned above that each vector space $\mathrm{CB}_n(A) = A^{\otimes(n+2)}$ is an X-module. Using the module-algebra condition (2.2.1), it is straightforward to see that each ∂_j is X-linear. Therefore, $\mathrm{CB}_*(A)$ can be regarded as a chain complex of X-modules.

Note that in the case $H = K$, the chain complex $\mathrm{CB}_*(A)$ of $A \otimes A^{\mathrm{op}}$-modules is the usual bar complex of A.

2.5. **Hopf-Hochschild cochain complex.** The *Hopf-Hochschild cochain complex of A with coefficients in A* is the cochain complex of vector spaces:

$$(\mathrm{CH}^*_{\mathrm{Hopf}}(A,A), d_{\mathrm{CH}}) \;\stackrel{\mathrm{def}}{=}\; \mathrm{Hom}_X((\mathrm{CB}_*(A), d^{\mathrm{CB}}), A). \tag{2.5.1}$$

Its nth cohomology module, denoted by $HH^n_{\mathrm{Hopf}}(A,A)$, is called the *$n$th Hopf-Hochschild cohomology of A with coefficients in A.*

When $H = K$, the cochain complex $(\mathrm{CH}^*_{\mathrm{Hopf}}(A,A), d_{\mathrm{CH}})$ is the usual Hochschild cochain complex of A with coefficients in itself [11], and $HH^n_{\mathrm{Hopf}}(A,A)$ is the usual Hochschild cohomology module.

In what follows, we will use the notation $\mathrm{CH}^*_{\mathrm{Hopf}}(A,A)$ to denote (i) the Hopf-Hochschild cochain complex $(\mathrm{CH}^*_{\mathrm{Hopf}}(A,A), d_{\mathrm{CH}})$, (ii) the sequence $\{\mathrm{CH}^n_{\mathrm{Hopf}}(A,A)\}$ of vector spaces, or (iii) the graded vector space $\oplus_n \mathrm{CH}^n_{\mathrm{Hopf}}(A,A)$. It should be clear from the context what $\mathrm{CH}^*_{\mathrm{Hopf}}(A,A)$ means.

3. A

The purpose of this section is to show that the vector spaces $\mathrm{CH}^*_{\mathrm{Hopf}}(A,A)$ in the Hopf-Hochschild cochain complex of an H-module-algebra A with self coefficients has the structure of an operad with multiplication.

3.1. **Operads.** Recall from [16, 17] that an *operad* $\mathcal{O} = \{\mathcal{O}(n), \gamma, \mathrm{Id}\}$ consists of a sequence of vector spaces $\mathcal{O}(n)$ ($n \geq 1$) together with structure maps

$$\gamma\colon \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \;\to\; \mathcal{O}(n_1 + \cdots + n_k), \tag{3.1.1}$$

for $k, n_1, \ldots, n_k \geq 1$ and an *identity element* $\mathrm{Id} \in \mathcal{O}(1)$, satisfying the following two axioms.

(1) The structure maps γ are required to be *associative*, in the sense that

$$\begin{aligned}\gamma(\gamma(f; g_{1,k}); h_{1,N}) = \; & \gamma(f; \gamma(g_1; h_{1,N_1}), \ldots, \\ & \gamma(g_i; h_{N_{i-1}+1,N_i}), \ldots, \gamma(g_k; h_{N_{k-1}+1,N_k})).\end{aligned} \tag{3.1.2}$$

Here $f \in \mathcal{O}(k)$, $g_i \in \mathcal{O}(n_i)$, $N = n_1 + \cdots + n_k$, and $N_i = n_1 + \cdots + n_i$. Given elements $x_i, x_{i+1}, \ldots$, the symbol $x_{i,j}$ is the abbreviation for the sequence $x_i, x_{i+1}, \ldots, x_j$ or $x_i \otimes \cdots \otimes x_j$ whenever $i \leq j$.

(2) The identity element $\mathrm{Id} \in \mathcal{O}(1)$ is required to satisfy the condition that the linear map

$$\gamma(-; \mathrm{Id}, \ldots, \mathrm{Id})\colon \mathcal{O}(k) \to \mathcal{O}(k) \tag{3.1.3}$$

is equal to the identity map on $\mathcal{O}(k)$ for each $k \geq 1$.

What is defined above is usually called a *non-Σ operad* in the literature.

3.2. **Operad with multiplication.** Let $\mathcal{O}$ be an operad. A *multiplication* on $\mathcal{O}$ [10, Section 1.2] is an element $m \in \mathcal{O}(2)$ that satisfies

$$\gamma(m; m, \mathrm{Id}) \;=\; \gamma(m; \mathrm{Id}, m). \tag{3.2.1}$$

In this case, $(\mathcal{O}, m)$ is called an *operad with multiplication*.

3.3. **Operad with multiplication structure on $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$.** In what follows, in order to simplify the typography, we will sometimes write $\mathcal{C}(n)$ for the vector space $\mathrm{CH}^n_{\mathrm{Hopf}}(A, A)$. To show that the vector spaces $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ form an operad with multiplication, we first define the structure maps, the identity element, and the multiplication.

Structure maps: For $k, n_1, \ldots, n_k \geq 1$, define a map

$$\gamma\colon \mathcal{C}(k) \otimes \mathcal{C}(n_1) \otimes \cdots \otimes \mathcal{C}(n_k) \to \mathcal{C}(N) \tag{3.3.1}$$

by setting

$$\begin{aligned}&\gamma(f; g_{1,k})(a_{0,N+1}) \\ &\quad \overset{\text{def}}{=} f\left(a_0 \otimes g_1(1 \otimes a_{1,n_1} \otimes 1) \otimes \cdots \otimes g_i(1 \otimes a_{N_{i-1}+1,N_i} \otimes 1) \otimes \cdots \otimes a_{N+1}\right).\end{aligned} \tag{3.3.2}$$

Here the notations are as in the definition of an operad above, and each $a_i \in A$.

Identity element: Let $\mathrm{Id} \in \mathcal{C}(1)$ be the element such that

$$\mathrm{Id}(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 a_2. \tag{3.3.3}$$

This is indeed an element of $\mathcal{C}(1)$, since the identity map on A is H-linear.

Multiplication: Let $\pi \in \mathcal{C}(2)$ be the element such that

$$\pi(a_0 \otimes a_1 \otimes a_2 \otimes a_3) = a_0 a_1 a_2 a_3. \tag{3.3.4}$$

This is indeed an element of $\mathcal{C}(2)$, since the multiplication map $A^{\otimes 2} \to A$ on A is H-linear.

Theorem 3.4. *The data $\mathcal{C} = \{\mathcal{C}(n), \gamma, \mathrm{Id}\}$ forms an operad. Moreover, $\pi \in \mathcal{C}(2)$ is a multiplication on the operad $\mathcal{C}$.*

Proof. It is immediate from (3.3.2) and (3.3.3) that $\gamma(-; \mathrm{Id}^{\otimes k})$ is the identity map on $\mathcal{C}(k)$ for each $k \geq 1$.

To prove associativity of γ, we use the notations in the definition of an operad and compute as follows:

$$\begin{aligned}
&\gamma(\gamma(f; g_{1,k}); h_{1,N})(a_0 \otimes \cdots \otimes a_{M+1}) \\
&= \gamma(f; g_{1,k})(a_0 \otimes \cdots \otimes h_j(1 \otimes a_{M_{j-1}+1, M_j} \otimes 1) \otimes \cdots \otimes a_{M+1}) \\
&= f(a_0 \otimes \cdots \otimes g_i(1 \otimes z_i \otimes 1) \otimes \cdots \otimes a_{M+1}) \\
&= \gamma(f; \ldots, \gamma(g_i; h_{N_{i-1}+1, N_i}), \ldots)(a_0 \otimes \cdots \otimes a_{M+1}).
\end{aligned} \tag{3.4.1}$$

Here the element z_i $(1 \leq i \leq k)$ is given by

$$\begin{aligned}
z_i &= \bigotimes_{l=N_{i-1}+1}^{N_i} h_l(1 \otimes a_{M_{l-1}+1, M_l} \otimes 1) \\
&= h_{N_{i-1}+1}(1 \otimes a_{M_{N_{i-1}}+1, M_{N_{i-1}+1}} \otimes 1) \otimes \cdots \otimes h_{N_i}(1 \otimes a_{M_{N_i-1}+1, M_{N_i}} \otimes 1).
\end{aligned} \tag{3.4.2}$$

This shows that γ is associative and that $\mathcal{C} = \{\mathcal{C}(n), \gamma, \mathrm{Id}\}$ is an operad.

To see that $\pi \in \mathcal{C}(2)$ is a multiplication on $\mathcal{C}$, one observes that both $\gamma(\pi; \pi, \mathrm{Id})(a_0 \otimes \cdots \otimes a_4)$ and $\gamma(\pi; \mathrm{Id}, \pi)(a_0 \otimes \cdots \otimes a_4)$ are equal to the product $a_0 a_1 a_2 a_3 a_4$.

This finishes the proof of Theorem 3.4. ◻

4. B

The purpose of this section is to show that the graded vector space $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ admits the structure of a brace algebra with multiplication.

4.1. **Brace algebra.** For a graded vector space $V = \oplus_{n=1}^{\infty} V^n$ and an element $x \in V^n$, set $\deg x = n$ and $|x| = n - 1$. Elements in V^n are said to have *degree n*.

Recall from [10, Definition 1] that a *brace algebra* is a graded vector space $V = \oplus V^n$ together with a collection of brace operations $x\{x_1, \ldots, x_n\}$ of degree $-n$, satisfying the associativity axiom:

$$x\{x_{1,m}\}\{y_{1,n}\} = \sum_{0 \le i_1 \le \cdots \le i_m \le n} (-1)^{\varepsilon} x\{y_{1,i_1}, x_1\{y_{i_1+1,j_1}\}, y_{j_1+1}, \ldots, y_{i_m}, x_m\{y_{i_m+1,j_m}\}, y_{j_m+1,n}\}.$$

Here the sign is given by $\varepsilon = \sum_{p=1}^{m} \left(|x_p| \sum_{q=1}^{i_p} |y_q| \right)$.

4.2. **Brace algebra with multiplication.** Let $V = \oplus V^n$ be a brace algebra. A *multiplication* on V [10, Section 1.2] is an element $m \in V^2$ such that

$$m\{m\} = 0. \tag{4.2.1}$$

In this case, we call $V = (V, m)$ a *brace algebra with multiplication*.

4.3. **Brace algebra from operad.** Suppose that $\mathcal{O} = \{\mathcal{O}(n), \gamma, \mathrm{Id}\}$ is an operad. Define the following operations on the graded vector space $\mathcal{O} = \oplus \mathcal{O}(n)$:

$$x\{x_1, \ldots, x_n\} \overset{\text{def}}{=} \sum (-1)^{\varepsilon} \gamma(x; \mathrm{Id}, \ldots, \mathrm{Id}, x_1, \mathrm{Id}, \ldots, \mathrm{Id}, x_n, \mathrm{Id}, \ldots, \mathrm{Id}). \tag{4.3.1}$$

Here the sum runs over all possible substitutions of $x_1, \ldots, x_n$ into $\gamma(x; \ldots)$ in the given order. The sign is determined by $\varepsilon = \sum_{p=1}^{n} |x_p| i_p$, where i_p is the total number of inputs in front of x_p. Note that

$$\deg x\{x_1, \ldots, x_n\} = \deg x - n + \sum_{p=1}^{n} \deg x_p,$$

so the operation (4.3.1) is of degree $-n$.

Proposition 1 in [10] establishes that the operations (4.3.1) make the graded vector space $\oplus \mathcal{O}(n)$ into a brace algebra. Moreover, a multiplication on the operad $\mathcal{O}$ in the sense of §3.2 is equivalent to a multiplication on the brace algebra $\oplus \mathcal{O}(n)$. In fact, for an element $m \in \mathcal{O}(2)$, one has that

$$m\{m\} = \gamma(m; m, \mathrm{Id}) - \gamma(m; \mathrm{Id}, m). \tag{4.3.2}$$

It follows that the condition (3.2.1) is equivalent to (4.2.1). In other words, an operad with multiplication $(\mathcal{O}, m)$ gives rise to a brace algebra with multiplication $(\oplus \mathcal{O}(n), m)$. Combining this discussion with Theorem 3.4, we obtain the following result.

Corollary 4.4. *The graded vector space* $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ *is a brace algebra with brace operations as in* (4.3.1) *and multiplication* π (3.3.4)*.*

The brace operations on $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ can be described more explicitly as follows. For $f \in \mathcal{C}(k)$ and $g_i \in \mathcal{C}(m_i)$ $(1 \leq i \leq n)$, we have

$$f\{g_1, \ldots, g_n\} = \sum (-1)^{\varepsilon} \gamma(f; \mathrm{Id}^{r_1}, g_1, \mathrm{Id}^{r_2}, g_2, \ldots, \mathrm{Id}^{r_n}, g_n, \mathrm{Id}^{r_{n+1}}), \tag{4.4.1}$$

where $\mathrm{Id}^r = \mathrm{Id}^{\otimes r}$. Here the r_j are given by

$$r_j = \begin{cases} i_1 & \text{if } j = 1, \\ i_j - i_{j-1} - 1 & \text{if } 2 \leq j \leq n, \\ k - i_n - 1 & \text{if } j = n + 1, \end{cases} \tag{4.4.2}$$

and

$$\varepsilon = \sum_{p=1}^{n} (m_p - 1) i_p. \tag{4.4.3}$$

Write $M = \sum_{i=1}^n m_i$ and $M_j = \sum_{i=1}^j m_i$. Then for an element $a_{0,\, k+M-n+1} \in A^{\otimes (k+M-n)}$, we have

$$\begin{aligned} f\{g_{1,n}\}(a_{0,\, k+M-n+1}) = \sum (-1)^{\varepsilon} f(a_{0, i_1} &\otimes g_1(1 \otimes a_{i_1+1,\, i_1+m_1} \otimes 1) \otimes \cdots \\ &\otimes a_{i_{j-1}+M_{j-1}-(j-1)+2,\, i_j+M_{j-1}-j+1} \\ &\otimes g_j(1 \otimes a_{i_j+M_{j-1}-j+2,\, i_j+M_j-j+1} \otimes 1) \otimes \cdots \\ &\otimes a_{i_n+M-n+2,\, k+M-n+1}). \end{aligned} \tag{4.4.4}$$

5. H G

The purpose of this section is to observe that the brace algebra with multiplication structure on $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ induces a homotopy Gerstenhaber algebra structure.

5.1. **Homotopy G-algebra.** Recall from [10, Definition 2] that a *homotopy G-algebra* $(V, d, \cup)$ consists of a brace algebra $V = \oplus V^n$, a degree $+1$ differential d, and a degree 0 associative $\cup$-product that make V into a differential graded algebra, satisfying the following two conditions.

(1) The $\cup$-product is required to satisfy the condition

$$(x_1 \cup x_2)\{y_{1,n}\} = \sum_{k=0}^{n} (-1)^{\varepsilon} x_1\{y_{1,k}\} \cup x_2\{y_{k+1,n}\},$$

where $\varepsilon = |x_2| \sum_{p=1}^k |y_p|$, for $x_i, y_j \in V$.

(2) The differential is required to satisfy the condition

$$
\begin{aligned}
& d(x\{x_{1,n+1}\}) - (dx)\{x_{1,n+1}\} \\
& \quad - (-1)^{|x|} \sum_{i=1}^{n+1} (-1)^{|x_1|+\cdots+|x_{i-1}|} x\{x_1, \ldots, dx_i, \ldots, x_{n+1}\} \\
= \; & (-1)^{|x||x_1|+1} x_1 \cup x\{x_{2,n+1}\} \\
& \quad + (-1)^{|x|} \sum_{i=1}^{n} (-1)^{|x_1|+\cdots+|x_{i-1}|} x\{x_1, \ldots, x_i \cup x_{i+1}, \ldots, x_{n+1}\} \\
& \quad - x\{x_{1,n}\} \cup x_{n+1}.
\end{aligned}
$$

5.2. **Homotopy G-algebra from brace algebra with multiplication.** A brace algebra with multiplication $V = (V, m)$ gives rise to a homotopy G-algebra $(V, d, \cup)$ [10, Theorem 3], where the $\cup$-product and the differential are defined as:

$$
\begin{aligned}
x \cup y & \overset{\text{def}}{=} (-1)^{\deg x} m\{x, y\}, \\
d(x) & \overset{\text{def}}{=} m\{x\} - (-1)^{|x|} x\{m\}.
\end{aligned}
\tag{5.2.1}
$$

This applies to the brace algebra $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ with multiplication π (Corollary 4.4).

Corollary 5.3. *For an H-module-algebra A, $\mathcal{C} = (\mathrm{CH}^*_{\mathrm{Hopf}}(A, A), d, \cup)$ is a homotopy G-algebra.*

5.4. **Comparing differentials.** At this moment, there are two differentials on the graded vector space $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$, namely, the differential d^n (5.2.1) induced by the multiplication π and the Hopf-Hochschild differential d^n_{CH}. The following result ensures that the cohomology modules defined by these two differentials are the same.

Theorem 5.5. *The equality $d^n_{\mathrm{CH}} = (-1)^{n+1} d^n$ holds for each n.*

Proof. Pick $f \in \mathrm{CH}^n_{\mathrm{Hopf}}(A, A)$. Then we have

$$
\begin{aligned}
d^n f & = \pi\{f\} + (-1)^n f\{\pi\} \\
& = (-1)^{n-1} \gamma(\pi; \mathrm{Id}, f) + \gamma(\pi; f, \mathrm{Id}) \\
& \quad + (-1)^n \sum_{i=1}^{n} (-1)^{i-1} \gamma(f; \mathrm{Id}^{i-1}, \pi, \mathrm{Id}^{n-i}).
\end{aligned}
\tag{5.5.1}
$$

It follows that

$$
(-1)^{n+1} d^n f = \gamma(\pi; \mathrm{Id}, f) + (-1)^{n+1} \gamma(\pi; f, \mathrm{Id}) + \sum_{i=1}^{n} (-1)^i \gamma(f; \mathrm{Id}^{i-1}, \pi, \mathrm{Id}^{n-i}). \tag{5.5.2}
$$

Observe that [14, page 8]

$$
g(a_{0,n+1}) = a_0 g(1 \otimes a_{1,n} \otimes 1) a_{n+1} \tag{5.5.3}
$$

for $g \in \mathrm{CH}^n_{\mathrm{Hopf}}(A, A)$. Using (5.5.3) and applying the various terms in (5.5.2) to an element $a_{0,n+2} \in \mathrm{CB}_{n+1}(A) = A^{\otimes(n+3)}$, we obtain

$$\begin{aligned} \gamma(\pi; \mathrm{Id}, f)(a_{0,n+2}) &= f(a_0 a_1 \otimes a_{2,n+2}), \\ \gamma(\pi; f, \mathrm{Id})(a_{0,n+2}) &= f(a_{0,n} \otimes a_{n+1} a_{n+2}), \\ \gamma(f; \mathrm{Id}^{i-1}, \pi, \mathrm{Id}^{n-i})(a_{0,n+2}) &= f(a_{0,i-1} \otimes a_i a_{i+1} \otimes a_{i+2,n+2}). \end{aligned} \tag{5.5.4}$$

The Theorem now follows immediately from (5.5.2) and (5.5.4). ◻

Corollary 5.6. *There is an isomorphism of cochain complexes*

$$(\mathrm{CH}^*_{\mathrm{Hopf}}(A, A), d_{\mathrm{CH}}) \cong (\mathrm{CH}^*_{\mathrm{Hopf}}(A, A), d).$$

Moreover, the cohomology modules on $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ *defined by the differentials* d_{CH} *and* d *are equal.*

6. G

The purpose of this section is to observe that the homotopy G-algebra structure on $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ gives rise to a G-algebra structure on the Hopf-Hochschild cohomology modules $HH^*_{\mathrm{Hopf}}(A, A)$.

6.1. **Gerstenhaber algebra.** Recall from [10, Section 2.2] that a *G-algebra* $(V, \cup, [-,-])$ consists of a graded vector space $V = \oplus V^n$, a degree 0 associative $\cup$-product, and a degree -1 graded Lie bracket

$$[-,-] \colon V^m \otimes V^n \to V^{m+n-1},$$

satisfying the following two conditions:

$$\begin{aligned} x \cup y &= (-1)^{\deg x \deg y} y \cup x, \\ [x, y \cup z] &= [x, y] \cup z + (-1)^{|x| \deg y} y \cup [x, z]. \end{aligned} \tag{6.1.1}$$

In other words, the $\cup$-product is graded commutative, and the Lie bracket is a graded derivation for the $\cup$-product. In particular, a G-algebra is a graded version of a Poisson algebra. This algebraic structure was first studied by Gerstenhaber [8].

6.2. ***G*-algebra from homotopy *G*-algebra.** If $(V, d, \cup)$ is a homotopy G-algebra, one can define a degree -1 operation on V,

$$[x, y] \overset{\mathrm{def}}{=} x\{y\} - (-1)^{|x||y|} y\{x\}. \tag{6.2.1}$$

Passing to cohomology, $(H^*(V, d), \cup, [-,-])$ becomes a G-algebra ([10] Corollary 5 and its proof).

Combining the previous paragraph with Corollary 5.3 and Corollary 5.6, we obtain the following result.

Corollary 6.3. *The Hopf-Hochschild cohomology modules $HH^*_{\text{Hopf}}(A, A)$ of an H-module-algebra A admits the structure of a G-algebra.*

This G-algebra can be described on the cochain level more explicitly as follows. Pick $\varphi \in \mathrm{CH}^n_{\text{Hopf}}(A, A)$ and $\psi \in \mathrm{CH}^m_{\text{Hopf}}(A, A)$. Then

$$\begin{aligned} (\psi \cup \varphi)(a_{0,m+n+1}) &= (-1)^{m+n-1}\psi(a_{0,m} \otimes 1)\varphi(1 \otimes a_{m+1,m+n+1}), \\ [\psi, \varphi] &= \psi\{\varphi\} - (-1)^{(m-1)(n-1)}\varphi\{\psi\}, \end{aligned} \tag{6.3.1}$$

where, writing $a = a_{0,m+n}$,

$$\begin{aligned} \psi\{\varphi\}(a) &= \sum_{i=1}^{m} (-1)^{(i-1)(n-1)}\psi(a_{0,i-1} \otimes \varphi(1 \otimes a_{i,i+n-1} \otimes 1) \otimes a_{i+n,m+n}), \\ \varphi\{\psi\}(a) &= \sum_{j=1}^{n} (-1)^{(j-1)(m-1)}\varphi(a_{0,j-1} \otimes \psi(1 \otimes a_{j,j+m-1} \otimes 1) \otimes a_{j+m,m+n}). \end{aligned} \tag{6.3.2}$$

In particular, if $m = n = 1$, then the bracket operation

$$[\psi, \varphi](a_{0,2}) = \psi(a_0 \otimes \varphi(1 \otimes a_1 \otimes 1) \otimes a_2) - \varphi(a_0 \otimes \psi(1 \otimes a_1 \otimes 1) \otimes a_2) \tag{6.3.3}$$

gives $HH^1_{\text{Hopf}}(A, A)$ a Lie algebra structure. There is another description of this Lie algebra in terms of (inner) derivations in [14, Proposition 3.9].

7. D ’ C -

The purpose of this section is to observe that a version of Deligne’s Conjecture holds for the Hopf-Hochschild cochain complex of a module-algebra. The original Deligne’s Conjecture for Hochschild cohomology is as follows.

Deligne’s Conjecture ([6])**.** *The Hochschild cochain complex $C^*(R, R)$ of an associative algebra R is an algebra over a suitable chain model of May’s little squares operad $\mathcal{C}_2$* [16].

A positive answer to Deligne’s conjecture was given by, among others, McClure and Smith [18, Theorem 1.1] and Kaufmann [13, Theorem 4.2.2]. There is an operad $\mathcal{H}$ whose algebras are the brace algebras with multiplication (§4.2). For an associative algebra R, the Hochschild cochain complex $C^*(R, R)$ is a brace algebra with multiplication and hence an $\mathcal{H}$-algebra. McClure and Smith showed that $\mathcal{H}$ is quasi-isomorphic to the chain operad $\mathcal{S}$ obtained from the little squares operad $\mathcal{C}_2$ by applying the singular chain functor, thereby proving Deligne’s Conjecture.

It has been observed that the Hopf-Hochschild cochain complex $\mathrm{CH}^*_{\text{Hopf}}(A, A)$ is a brace algebra with multiplication (Corollary 4.4). Therefore, we can use the result of McClure and Smith [18, Theorem 1.1] to obtain the following version of Deligne’s Conjecture for module-algebras.

Corollary 7.1 (Deligne's Conjecture for module-algebras)**.** *The Hopf-Hochschild cochain complex* $\mathrm{CH}^*_{\mathrm{Hopf}}(A, A)$ *of an H-module-algebra A is an algebra over the McClure-Smith operad* $\mathcal{H}$ *that is a chain model for the little squares operad* $\mathcal{C}_2$.

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